A Mathematical Note

Functional Inversion Strategies

with emphasis on their application to

Inversion Problems Posed by Napier, Lambert & Sommerfeld

Nicholas Wheeler
August 2017

Introduction.[1] The functional inversion problem, stripped of all nuance, is very easily posed: given that $y = f(x)$, display a function $g(y)$ such that

$$g(y) = g(f(x)) = x$$

But nuance—and its attendant complications— are in many instances the name of the game. If $f(x)$, taken here to be a real-valued function of a real variable, is continuous and monotonic then the form and essential properties of $g(y)$—if not its analytic description—can be read off from a graph of $y = f(x)$. From such graphs one can gain also a sense of the complications that arise if $f(x)$ is discontinuous, or not monotonic; it becomes obvious that $g(y)$ will in many cases be multi-valued.

The solution of inversion problems—like the evaluation of integrals (and sometimes *because* it involves the evaluation of integrals)—often entails expansion of the list of previously "named functions." Historically, one of the first such events had to do with the invention of $\log_b y$.

John Napier (1550–1617), who had in mind mainly the practical needs of astronomers and navigators, is credited[2] for having been the first to recognize the computational power latent in the "law of exponents," (which itself goes back to Archimedes). If $y_1 = b^{x_1}$ and $y_2 = b^{x_2}$ then $y_1 \cdot y_2 = b^{x_1+x_2}$, and multiplication has been reduced to addition.[3] Napier initially called the numbers

[1] Work on this revision of some earlier drafts was begun on 22 August, the day after a celestial inversion event of awesome beauty, the Great American Eclipse of 2017.

[2] Not quite accurately; Jost Bürgi (1552–1632), a Swiss watch and instrument maker, had the same idea, independently and almost simultaneously.

[3] In previous decades people had used identities such as

$$cos\alpha \cdot \cos\beta = \tfrac{1}{2}\{\cos(\alpha - \beta) + \cos(\alpha + \beta)\}$$

to accomplish that same objective. Success here presupposes the availability (motivates the construction) of trig tables.

$\{x_1, x_2\}$ "artificial numbers," and later "logarithms" (from *logos* + *arithmos* = "ratio + numbers"). Standardly we write $x_i = \log_b y_i$, which we call the "log of y_i to base b." How Napier arrived—without the assistance of the calculus—at the "Napierian logarithm" (subsequently called the "natural logarithm")—how, in short, he arrived at the invention of e—is a tangled tale told in Chapter 4 of Julian Havil's *John Napier: Life, Logarithms & Legacy* (2014).[4] Tangled also is the tale of how he proceeded to the *construction* of his log tables, work soon replicated all over Europe by mathematicians who appreciated the fundamental significance of Napier's creation. Descendents of those log tables served as indispensable aids to scientists and engineers until they were—like sliderules—rendered obsolete by computers in the 1970s.

Napier's logarithm became the "logarithmus naturalis" (terminology of Pietro Mengoli and Nicholas Mercator) even before the invention of the calculus, but it is the calculus (Euler, 1748) that lent vivid meaning to the adjective; from $\log_b(x) = \log x/\log b$ we have

$$b^x = 1 + x\log b + \tfrac{1}{2}(x\log b)^2 + \tfrac{1}{6}(x\log b)^3 + \tfrac{1}{24}(x\log b)^4 + \cdots$$
$$\tfrac{d}{dx}b^x = b^x \log b$$

both of which (by $\log e = 1$) are simplified by setting $b \to e$.

From Napier to Lambert to Sommerfeld. In 1758, Johann Lambert (1728–1777) had occasion to study an equation

$$x^\alpha - x^\beta = (\alpha - \beta)y\,x^{\alpha+\beta}$$

that has come to be known as "Lambert's transcendental equation," and which came to Euler's attention in 1764. In the limit $\alpha \to \beta$[5] this assumes the special form

$$\log x = y\,x^\beta$$

[4] Havil provides a fascinating account of the remarkably rich life of Napier, of whom Mark Napier (1834) wrote: *With the exception of those little episodes we have noticed, of battle, murder and sudden death, Popish plots, pestilence and famine, ever and anon demanding more or less of our philosopher's time and attention; together with the whole charge of his twelve children, and more than half the charge of his unruly brothers, besides farming operations extending from the Firth to the banks of the Teith, and the islands of Lochlomond; mingled with ocassional demands for his "singular judgment," from the General Assembly of the church, to the dark outlaw who indulged in magic, and the courtly lawyer to sought a lesson in mesuration; with the exception, we say, of these inevitable interruptions, our philosopher lived the life of an intellectual hermit, entirely devoted to his theological and mathematical speculations, and delighting in no converse so much as the clear crow of his favorite bird, more powerful, to "dismiss the demons" than all the incantations of Lilly.*

[5] Use

$$\lim_{\alpha\to\beta}\frac{y^\alpha - y^\beta}{\alpha - \beta} = y^\beta \log y$$

which was studied by Euler in 1783. The notational adjustment $\beta \to k^{-1}$ produces

$$y = x^k \log x \tag{1}$$

which in the case $k = 0$ reads

$$y = \log x \quad \Longleftrightarrow \quad x = e^y \quad \text{by standard functional inversion} \tag{2}$$

and in the case $k = 1$ becomes precisely the equation

$$y = x \log x \tag{3}$$

that sparked this entire discussion: the complex extension of the $y(x)$ produced by functional inversion of (3) is central to a 1899 paper by Arnold Sommerfeld (who at the time was unacquainted with the then-long history of what we have called "Sommerfeld's equation"), of which Robert Warnock has provided an English translation.[6]

Basic inversion methods.

$\boxed{\text{ALGEBRAIC INVERSION}}$

Look to the case

$$y = f(x) \equiv 4 + 2x + x^2$$

which (Figure 1f) describes an up-turned parabola with vertex (turning point) at $\{x_0, y_0\}$, where $x_0 = -1$ and $y_0 = f(x_0) = 3$. It is evident from the figure that

- $f(x)$ is monotone decreasing for $x < x_0$, monotone increasing for $x > x_0$;
- $g(y)$ is single-valued at $y = y_0$, double-valued for $y > y_0$.

Algebra supplies

$$x = g(y) = -1 \pm \sqrt{y - 3} \equiv g_\pm(y)$$

which are complex for $y < y_0$, real-valued for $y > y_0$. From

$$g_+(y > y_0) > x_0 \quad \text{and} \quad g_-(y > y_0) < x_0$$

we see that

$$y = f(x) \text{ is inverted by } \begin{cases} g_+(y) \text{ for } x\text{-values } > x_0 \\ g_-(y) \text{ for } x\text{-values } < x_0 \end{cases}$$

The functions $g_\pm(y)$, when graphed (Figure 1g), fuse to produce to produce a continuous curve, a parabola that opens to the right, with vertex at $\{y^0, x^0\}$ where $y^0 = 3, x^0 = -1$.

Similar in all essential respects is the Gaussian case

$$y = f(x) \equiv e^{-x^2}$$

which (Figure 2f) is monotone increasing/decreasing according as $x \lessgtr x_0 = 0$.

---

[6] "On the propagation of electrodynamic waves along a wire" (unpublished).

Here y ranges on the unit interval $[0,1]$, and within that interval $g(y)$ is clearly double-valued. Immediately

$$x = g(y) = \pm\sqrt{\log(1/y)} \equiv g_\pm(y)$$

$$y = f(x) \text{ is inverted by } \begin{cases} g_+(y) \text{ for } x\text{-values } > x_0 \\ g_-(y) \text{ for } x\text{-values } < x_0 \end{cases}$$

The functions $g_\pm(y)$, when graphed (Figure 2g), fuse to produce to produce a continuous curve, a Gaussian that peaks to the right.

Those examples illustrate this general point: the function $f(x)$ can be expected in typical cases to partition the x-axis into disjoint "monotonicity intervals" $I_n : x_n < x < x_{n+1}$ bounded by points where $f(x)$ is either level or discontinuous, and within each of which $f(x)$ is monotonic (Figure 3). The functional inverse $g(y)$ of $f(x)$ is typically multivalued, acquiring distinct forms $g_n(y)$ on each of the intervals:

$$y = f(x) \text{ is inverted by } g_n(y) \text{ for } x \in I_n$$

INVERSION BY INTEGRATION

Only seldom can inversion problems be solved by such direct algebraic means (consider the case $y = f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5$). An alternative of which we will have occasion to make use proceeds from the basic relation $f(g(y)) - y = 0$. Differentiation supplies a diffential equation

$$f'(g(y)) \cdot g'(y) - 1 = 0 \tag{4}$$

which can in favorable cases be integrated. Look again, for example, to the quadratic case $f(x) = 4 + 2x + x^2$, which gives

$$2(g(y) + 1)g'(y) - 1 = 0$$

whence $\int(2g + 2)dg = \int dy$. So we have $g^2 + 2g = y + c$, or

$$g(y) = -1 \pm \sqrt{y + 1 + c}$$

Those functions coincide at $y = -(1 + c)$, where

$$f(x) = 4 + 2x + x^2 = -(1 + c)$$

entails

$$x = -1 \pm \sqrt{-4 - c}$$

which also coincide (as inversion requires) if and only if $c = -4$, and their coincident value is $x_0 = -1$, where $f(x_0) \equiv y_0 = 3$. Thus do we recover

$$g_\pm(y) = -1 \pm \sqrt{y - 3}$$

and reproduce the vertex details shown in Figures 1.

Of greater interest, both historically and methodologically, is Napier's case

$$f(x) = b^x$$

Working now from $\frac{d}{dx}\big[g(f(x)) - x = 0\big]$ we have

$$\frac{dg(f)}{df} = \Big[\frac{df(x)}{dx}\Big]^{-1}$$

In the present instance

$$\frac{df(x)}{dx} = \tfrac{d}{dx}b^x = Kb^x = Kf(x) \quad \text{with} \quad K(b) = \log_e b$$

so from $dg = [Kf]^{-1}df$—which gives

$$\int_0^{g(f(x))} dg = \frac{1}{K}\int_1^x \frac{df}{f}$$

(here the lower limits arise from $f(0) = 1$)—we obtain

$$g(b^x) = \frac{1}{K(b)}\int_1^x \frac{dt}{t} \text{ , to which Napier gave the name “}\log_b x\text{”} \tag{5.1}$$

The special simplicy of the "natural" (originally "Napierian") logarithm arises from the circumstance that $K(e) = \log_e e = 1$. In the special case $f(x) = e^x$ we have

$$g(e^x) = \int_1^x \frac{dt}{t} \text{ , an intractable integral given the name “}\log x\text{”} \tag{5.2}$$

In this notation (5.1) becomes

$$\log_b x = \frac{\log x}{\log b}$$

Basic properties of $\log x$ follow directly from the integral representation (5.2), (see Figure 4).

Inversion problems, approached in this way, quite commonly result in intractable integrals to which one is obliged simply to assign names, and to grant admission to the canon of "higher functions." Examples are

$$\text{“arctan } x\text{”} = \int_0^x \frac{dt}{1+t^2}$$

$$\text{“arcsin } x\text{”} = \int_0^x \frac{dt}{\sqrt{1-t^2}}$$

The integral encountered in (5) stands in obviously close relationship to the harmonic series

$$H_n = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \cdots + \tfrac{1}{n}$$

Indeed, one has

$$H_n - 1 < \log n < H_n - \tfrac{1}{n}$$

and—most importantly—

$$\begin{aligned}\lim_{n\to\infty}\big[H_n - \log n\big] &= \gamma,\ \text{the Euler-Mascheroni constant}\\ &= 0.5772156649\end{aligned}$$

The story is (once again) nicely told—so far as it can be told; essential properties of γ remain unknown—by Julian Havil.[7]

LAGRANGE INVERSION

Of more general utility—available whenever $f(x)$ can be developed as a power series

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots = \sum_{n=0}^{\infty} a_nx^n$$

—is the Lagrange inversion formula[8] (1770, 1773), which displays the inverse function $g(y)$ as a power series in $y - a$. The *Mathematica* command[9]

```
InverseSeries[Series[f[y],{y,0,n}]]
```

produces the first n terms of that series; thus (set $n = 5$)

$$\begin{aligned}g_5(y) = \frac{1}{a_1}(y-a_0) - \frac{a_2}{a_1^3}(y-a_0)^2 + \frac{2a_2^2 - a_1a_2}{a_1^5}(y-a_0)^3\\ + \frac{-5a_2^3 + 5a_1a_2a_3 - a_1^2a_4}{a_1^7}(y-a_0)^4\\ + \frac{14a_2^4 - 21a_1a_2^2a_3 + 3a_1^2a_3^2 + 6a_1^2a_2a_4 - a_1^3a_5}{a_1^9}(y-a_0)^5\end{aligned}$$

Increasing the value of n brings additional terms into play, but does not alter the value of the terms displayed above. *Mathematica*-assisted calculation gives

$$g_5\Big(\sum_{n=0}^{m} a_nx^n\Big) = x + \text{terms of orders } x^{6=5+1} \text{ through } x^{5m}$$

which we read as indication that in the limit $g_\infty(\sum_{n=0}^\infty a_nx^n) = x$.

---

[7] See his *Gamma: Exploring Euler's Constant* (2003), esapecially Chapter Five: "Gamma's Birthplace."

[8] Sometimes called the "Lagrange-Bürmann inversion formula" because Lagrange's formula is a special case of the more general result developed by H. Bürmann in 1799.

[9] The command is based upon general formulæ that can be found in the literature, but are in themselves far too complicated to be useful except in favorable cases.

For the truncated quadratic series $f(x) = a+bx+cx^2$ the Lagrange formula ($n = 7$) produces

$$\begin{aligned} g(y) = \frac{y-a}{b} - \Big(\frac{c}{b}\Big)\Big(\frac{y-a}{b}\Big)^2 + 2\Big(\frac{c}{b}\Big)^2\Big(\frac{y-a}{b}\Big)^3 - 5\Big(\frac{c}{b}\Big)^3\Big(\frac{y-a}{b}\Big)^4 \\ + 14\Big(\frac{c}{b}\Big)^4\Big(\frac{y-a}{b}\Big)^5 - 42\Big(\frac{c}{b}\Big)^5\Big(\frac{y-a}{b}\Big)^6 + 132\Big(\frac{c}{b}\Big)^6\Big(\frac{y-a}{b}\Big)^7 + \cdots \end{aligned}$$

from which by computation we obtain a result

$$g_7(a+bx+cx^2) = x + \text{terms of orders } x^{8=7+1} \text{ through } x^{14=7\times 2}$$

that supports the assertion that $g_\infty(a+bx+cx^2) = x$. We know, however, that in this instance algebraic inversion supplies *two* inverse functions, namely

$$\begin{aligned} g_\pm(y) &= \frac{-b \pm \sqrt{b^2-4ac+4cy}}{2c} \\ &= \frac{-b\pm\sqrt{b^2}}{2c} \pm \sqrt{b^2}\Big\{\frac{y-a}{b^2} - c\Big(\frac{y-a}{b^2}\Big)^2 + 2c^2\Big(\frac{y-a}{b^2}\Big)^3 - 5c^3\Big(\frac{y-a}{b^2}\Big)^4 + \cdots \end{aligned}$$

The functions $f_\pm(y)$ are coincident at $y = (4ac-b^2)/4c$ but otherwise distinct. They are found to be well approximated by the preceding pair of series (and their distinctness preserved) only if all occurances of $\sqrt{b^2}$ are interpreted to mean b. But if all occurances of $\pm\sqrt{b^2}$ are interpreted to mean b then one recovers the solitary Lagrange series. Lagrange inversion has in this instance captured $g_+(y)$ but missed $g_-(y)$. We can expect similar failure whenever $g(y)$ is multi-valued.

Look now to the case $f(x) = e^x$. Lagrange inversion gives

$$\begin{aligned} g(y) &= (y-1) - \tfrac{1}{2}(y-1)^2 + \tfrac{1}{3}(y-1)^3 - \tfrac{1}{4}(y-1)^4 + \tfrac{1}{5}(y-1)^5 - \cdots \\ &= -\sum_{n=1}^\infty \frac{1}{n}\sum_{k=0}^n (-)^k \binom{n}{k} y^k \\ &= \sum_{k=0}^\infty \Big\{(-)^{k+1}\sum_{n=1}^\infty \frac{1}{n}\binom{n}{k}\Big\} y^k \\ &\equiv \sum_{k=0}^\infty G_k\, y^k \end{aligned}$$

Here $G_0 = -\sum_{n=1}^\infty \frac{1}{n}$ diverges harmonically, and indeed, all of the coefficients diverge: $G_k = \pm\infty$ according as k is odd or even. This is Lagrange's way of saying that $\log y$ cannot be developed as a power series. But because the Lagrange formula refers to the *formal* structure of series—irrespective of their convergence properties—we have

$$\texttt{InverseSeries[InverseSeries[Series[}e^x, \{x,0,5\}\texttt{]]]} \\ = 1 + x + \tfrac{1}{2!}x^2 + \tfrac{1}{3!}x^3 + \tfrac{1}{4!}x^4 + \tfrac{1}{5!}x^5 + O[x]^6 = e^x$$

Similarly, we in the case $f(x) = e^x - 1$ obtain

$$\begin{aligned} g(y) &= \texttt{InverseSeries[Series[}e^y - 1, \{y, 0, 5\}\texttt{]]} \\ &= y - \tfrac{1}{2}y^2 + \tfrac{1}{3}y^3 - \tfrac{1}{4}y^4 + \tfrac{1}{5}y^5 - \cdots \\ &= \log(y+1) \end{aligned}$$

and

$$\begin{aligned} &\texttt{InverseSeries[InverseSeries[Series[}e^x - 1, \{x, 0, 5\}\texttt{]]]} \\ &\qquad = x + \tfrac{1}{2!}x^2 + \tfrac{1}{3!}x^3 + \tfrac{1}{4!}x^4 + \tfrac{1}{5!}x^5 + O[x]^6 = e^x - 1 \end{aligned}$$

In this case non-convergence was not an issue; $g(y)$ does (for $|y| < 1$) converge, to a "named function," though it was here again only to the formal properties of the g-series that the Lagrange formula looked when reconstructing $f(x)$.

From log to Lambert. We advance now from $y = e^x$ to the "next simplest case"

$$y = f(x) \equiv x\,e^x = x + x^2 + \tfrac{1}{2!}x^3 + \tfrac{1}{3!}x^4 + \tfrac{1}{4!}x^5 + \tfrac{1}{5!}x^6 + \cdots \tag{6}$$

It is evident from the associated graph (Figure 5) that

$$g(y) \text{ is} \begin{cases} \text{double-valued for } y < 0 \\ \text{single-valued for } \ \ y > 0 \end{cases}$$

The equation $y = x\,e^x$ does not yield to direct algebraic inversion, but when written $x = y/e^x$ suggests the recursive process

$$x = y/\exp(y/\exp(y/\exp\,(y/\cdots)))$$

which in *Mathematica* can be implemented by defining $w(y, s) = y/\exp(s)$ and

$$\begin{aligned} W(y, n) &= \texttt{Nest[w[y,\#]\&,y,n]} \\ &= \text{result of } n^{\text{th}}\text{-order recursive iteration} \end{aligned}$$

The command `NestList[w[y,#]&,y,n]` can be used to check convergence, which obviously fails for $y < -1/e$, but

$$\begin{aligned} &W(-1/e + 0.00001, 100) = -0.979916 \\ &\qquad\qquad f(-0.979916) = (-1/e + 0.00001) + 0.000065 \end{aligned}$$

demonstrates that convergence is reasonably swift already for y-values only slightly above $y_{\text{min}} = -1/e$. 6-place stability is achieved with only 20 iterations for $-0.3 < y < 0.7$, but to achieve such stability for $y > 1.5$ the number n of iterations must be pushed to progressively higher values. Figure 6 shows the values assumed by $W(y, 20)$ at 100 equi-spaced points y_k, where

$$\begin{aligned} y_1 &= -1/e + 0.0186788 = -0.367879 \\ y_{100} &= 1.5 \end{aligned}$$

The scheme provides no hint of double-valuedness for $y < 0$ (but see below: page 13).

Differentiation of $g(y)e^{g(y)} - y = 0$ produces the differential equation

$$e^{g(y)}g'(y)[g(y)+1] - 1 = 0$$

for which the `DSolve` command produces

$$g(y) = \text{LambertW}(y+c)$$

We are informed by *Mathematica* that (as will emerge) $\text{LambertW}(0) = 0$, so to achieve $g(0) = 0$—forced by the equation which was our point of departure—we are obliged to set the constant of integration $c = 0$, giving

$$\begin{aligned} x = g(y) &= \text{LambertW}(y), \text{ henceforth denoted } W(y) \\ &= y - y^2 + \tfrac{3}{2}y^3 - \tfrac{8}{3}y^4 + \tfrac{125}{24}y^5 - \tfrac{56}{5}y^6 + \tfrac{16807}{720}y^7 - \cdots \qquad (7) \\ &= \sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!} y^n \\ &\equiv \sum_{n=1}^{\infty} G_n y^n \end{aligned}$$

We are gratified to discover that (7) is precisely the series produced by Lagrange inversion of (6). In those results there is again no hint of double-valuedness.

Giving names

$$\begin{aligned} f_5(x) &= \sum_{k=0}^{4} \frac{1}{k!} x^{k+1} \\ g_8(y) &= \sum_{n=1}^{8} \frac{(-n)^{n-1}}{n!} y^n \end{aligned}$$

to truncated expansions of $f(x)$ and $g(y)$, we find by computation that

$$g_8(f_5(x)) = x + \text{terms of orders } x^{6=5+1} \text{ through } x^{40=5*8}$$

which we read as indication that $g(y)$ is in fact the—or at least an—inverse of $f(x)$.

The coefficients G_n grow rapidly very large—in Stirling approximation

$$\begin{aligned} \log G_n &\approx (n-1)\log n - \left\{ \log\sqrt{2\pi} + (n+\tfrac{1}{2})\log n - n \right\} \\ &= n - \tfrac{3}{2}\log n - \log\sqrt{2\pi} \end{aligned}$$

so G_n grows a little more slowly than e^n —and for $y > 0$ successive terms in the series alternate in sign. Both of those circumstances impede convergence. Numerical experiments indicate that convergence becomes very slow and noisy as $|y|$ approaches 0.35. It will emerge that in fact the series converges if and only if

$$|y| < 1/e = 0.367879$$

The properties of "Lambert's W-function" $W(y)$ are treated in none of the standard higher-function handbooks, but the complex extension and diverse applications of which are developed in elaborate detail in a paper by Corless *at al*.[10]

The complex extension $W(z)$ of $W(x)$ is, like the complex extension $\log z$ of $\log x$, for typical values of z infinitely multivalued, which is to say: it possesses infinitely many sheets (marked by branch cuts). Those—signified $W_n(z)$, with $W_0(z) \equiv W(z)$—are produced by the *Mathematica* commands

$$\texttt{LambertW[n,z]} \quad : \quad n = 0,\, \pm1,\, \pm2,\, \ldots$$

and can be visualized by variants (replace `Im` by `Re`, `Abs`, `Arg`) of commands of the form[11]

```
Plot3D[Im[LambertW[n,x+iy]],{x,-2,2},{y,-1,1}]
```

From such figures (else from `Plot[LambertW[n,x],{x,-2,2}]`, which is blank unless $W_n(x)$ is real) we discover that

- $W_0(z)$ is real only on the semi-infinite real line $-(1/e) < x < \infty$;
- $W_1(z)$ is real only the real interval $-(1/e) < x < 0$;
- All other sheets are everywhere complex.

Graphs of $W_0(y)$ and $W_{-1}(y)$ are shown in Figure 7; they are seen to splice together to form a continuous curve, which for $y \in \{-1/e, 0\}$ is double-valued.[11]

Previously I inserted a truncation of the $x\,e^x$-series into a truncation of the $W_0(y)$-series to establish evidence that $x = W_0(y)$ is indeed the functional inverse of $y = x\,e^x$. There are, however, swifter and more instructive ways to achieve that same objective. *Mathematica* almost instantly supplies

$$\texttt{Series[LambertW}_0\texttt{[}x\,e^x\texttt{],\{x,0,10\}]} = x + O[x]^{11}$$

but $\texttt{Series[LambertW}_{-1}\texttt{[}x\,e^x\texttt{],\{x,0,10\}]}$ is a very long and unworkable mess, involving a great many `Log`, `Arg` and `Floor` operations. We can circumvent this difficulty by proceeding pointwise: the function $x\,e^x$ is (see Figure 5) double-valued for $x < 0$. It is found (use `FindRoot`) to acquire (say) the value

$$x\,e^x = -0.03 \quad \text{at} \quad \begin{cases} x = A \equiv -0.489402 \\ x = B \equiv -1.781340 \end{cases}$$

and we verify that indeed

$$\begin{aligned} W_0(-0.03) &= A \\ W_{-1}(-0.03) &= B \end{aligned}$$

---

[10] R. M. Corless, G. H. Gonnet, D. E. G. Hare, D. J. Jeffery & D. E. Knuth, "On the Lambert W function," Advances in Computational Mathematics **5**, 329–359 (1996).

[11] A possible point of confusion: To denote complex numbers I write $z = x + iy$. But in $W(y)$, the functional inverse of $y = x\,e^x$ (x real), the variable y is understood to be real; it does not refer to the imaginary part of anything.

On that evidence we conclude (and verify by additional test calculations) that (see Figure 7)

$$y = f(x) = x\,e^x \text{ is inverted by } \begin{cases} W_0(y) & \text{for } x\text{-values } > x_0 \\ W_-(y) & \text{for } x\text{-values } < x_0 \end{cases}$$
$$x_0 = -1$$

A brief note in passing: it is reported[12] that $W_0(y)$ admits of integral representation

$$W_0(y) = \frac{y}{\pi}\int_0^\pi \frac{(1-\nu\cot\nu)^2 + \nu^2}{y + \nu\csc\nu\cdot e^{-\nu\cot\nu}}\,d\nu \tag{8}$$

reminiscent of the representation (5.2) of $\log x$. *Mathematica* is unable to perform the symbolic integration, but the command

$$\frac{y}{\pi}\,\texttt{NIntegrate[}\frac{(1-\nu\cot\nu)^2 + \nu^2}{y + \nu\csc\nu\cdot e^{-\nu\cot\nu}}\texttt{,}\{\nu, 0, \pi\}\texttt{]} - \texttt{LambertW[}y\texttt{]}$$

returned values that did not exceed 10^{-10} when y ranged between $0.95(-1/e)$ and 4.0,[13] so (8) appears to be correct. The argument that led to (5.2) appears, however, to fail in this instance. Differentiating $W(f(x)) - x = 0$ with respect to x leads to

$$dW = \frac{df}{(x+1)e^x}$$

which is a dead end since the denominator cannot be expressed in terms of f.

Sommerfeld's inversion problem. As was remarked on page 3, Sommerfeld had physical reasons to be interested in the inversion of

$$y = f(x) = x\log x \tag{3}$$

which is plotted in Figure 8f, and from which we conclude that

$$g(y) \text{ is } \begin{cases} \text{double-valued for } -1/e < y < 0 \\ \text{single-valued for } \qquad\qquad y > 0 \end{cases}$$

A change of variable $x \to x(u) = e^u$ causes Sommerfeld's (3) to read

$$y = f(x(u)) = e^u u$$

which is precisely Lambert-Euler equation discussed in the preceding section, so immediately

$$g(y) = e^{W(y)}$$

[12] See the Wikipedia article "Lambert W function." This result does not appear in Corless.

[13] 234 equi-spaced points were tested. For small values of y the difference values did not exceed 10^{-11}.

or—more precisely—

$$\text{the inverse of } y = f(x) = x\log x \text{ is } \begin{cases} g_-(y) = \exp[W_{-1}(y)] & \text{if } 0 < x < 1/e \\ g_+(y) = \exp[W_0(y)] & \text{if } \quad x > 1/e \end{cases}$$

whch produces Figure 8g. Had we worked from

$$\tfrac{d}{dy}[g(y)\log g(y) - y] = g'(y)[1 + \log g(y)] - 1 = 0$$

the command `DSolve` would have produced

$$\begin{aligned} g(y) &= \frac{y+c}{W_0(y+c)} \\ &= \exp[W_0(y+c)] \text{ by a fundamental Lambert identity} \end{aligned}$$

but issued this well-advised WARNING: "Inverse functions are being used by Solve, so some solutions may not be found."

Lagrange inversion of

$$\exp[W_0(y)] = 1 + x - \tfrac{1}{2}x^2 + \tfrac{2}{3}x^3 - \tfrac{9}{8}x^4 + \tfrac{32}{15}x^5 - \tfrac{625}{144}x^6 + \cdots \tag{9}$$

gives what we recognize to be the series

$$\begin{aligned} f(x) = (x-1) &+ \tfrac{1}{2}(x-1)^2 - \tfrac{1}{6}(x-1)^3 + \tfrac{1}{12}(x-1)^4 - \tfrac{1}{20}(x-1)^5 \\ &+ \tfrac{1}{30}(x-1)^6 - \tfrac{1}{42}(x-1)^7 + \tfrac{1}{56}(x-1)^8 - \tfrac{1}{72}(x-1)^9 + \cdots \end{aligned}$$

that results when $x\log x$ is developed in powers of $(x-1)$.[14] The series (9) mimics the awkward features of the expansion (7) of $W_0(y)$. Unfortunately, a series analogous to (9) is—for the reason remarked on page 10—not available for $\exp[W_{-1}(y)]$.[15]

Sommerfeld—who in 1898 was ignorant of the work of Lambert-Euler (which by then lay 115 years in the past), who did not possess the W-analog of a log table, and for whom computer magic was not even a dream—stood in need of a way to calculate the x-values that lead via $y = x\log x$ to specified y-values, particularly in cases of the form $y = -u + iv$ where u and v are small positive real numbers. That there might exist *two* such x-values seems, though evident in Figure 8f, to have escaped his attention. Writing

$$x = \frac{y}{\log x}$$

he proposed to proceed by recursive iteration

[14] Signs alternate, denominators advance by addition of the next even integer, development in powers of x is not possible.

[15] One could, however, simulate one by fitting a high order polynomial to numerically-generated data points $\{y_k, W_{-1}(y_k)\}$.

$$x = y/\log(y/\log(y/\log(y/\log(\cdots)))) \tag{10.1}$$

though he might in that same spirit have written

$$x = \exp(y/x)$$

and proposed to proceed

$$x = \exp(y/\exp(y/\exp(y/\exp(\cdots)))) \tag{10.2}$$

From

$$\begin{aligned} p(y,s) &= y/\log s \\ q(y,t) &= \exp(y/t) \end{aligned}$$

I have constructed the commands

```
P[y_,n_]:=Nest[p[y,#]&,y,n]
PList[y_,n_]:=NestList[p[y,#]&,y,n]

Q[y_,n_]:=Nest[q[y,#]&,y,n]
QList[y_,n_]:=NestList[q[y,#]&,y,n]
```

Mathematica reports (set $n = 20$) that at $y = -0.1$ both processes converge rapidly, to

$$\begin{aligned} \texttt{P}[-0.1,20] &= 0.027955 \\ \texttt{Q}[-0.1,20] &= 0.894194 \end{aligned}$$

These are to be compared with

$$\begin{aligned} \exp[W_{-1}(-0.1)] &= 0.027955 \\ \exp[W_0(-0.1)] &= 0.894194 \end{aligned}$$

of which

$$f(0.027955) = f(0.894194) = -0.1$$

provide confirmation. But at $y = 0.1 > 0$ the picture changes, as from Figure 8g we expect it to do; convergence is still rapid in both cases, and we do have

$$\texttt{Q}[0.1,20] = 1.09557 = \exp[W_0(0.1)]\,, \quad f(1.09557) = 0.1$$

But

$$\begin{aligned} \texttt{P}[0.1,20] &= -0.0180768 - 0.0120544i \quad \begin{cases} \text{the sign of the imaginary} \\ \text{part oscillates} \end{cases} \\ \exp[W_{-1}(0.1)] &= -0.0127744 + 0.0124075i \end{aligned}$$

and at neither of those values does $f(\bullet) = 0.1$. We conclude that Sommerfeld's P-process reproduces the $g_-(y)$ branch of the inverse (which evidently answered to his physical requirements) while the Q-process reproduces the $g_+(y)$ branch.

Points produced by the P and Q-processes are shown in Figure 6. The recursive processes (10) were anticipated already on page 8; we are in position now to understand why—and to deal with the fact—that the process described there "provided no hint of double-valuedness." Sommerfeld had studied the convergence properties of the P-process already in 1898.[16]

One can only wonder whether the fact that P-process, with its iterated log functions, leads to a W-function has something to do with the circumstance that such functions are known as `ProductLog` functions to *Mathematica*, which recognizes `LambertW` only as a silent alias.

Generalized Lambert/Sommerfeld inversion. As was remarked on pages 2–3, Euler was led from Lambert's transcendental equation to an equation

$$y = x^k \log x \tag{1}$$

that gives back Sommerfeld's equation as a special case. Familiar manipulations (write $x = e^u$, then adjust the notation $u \to x$) bring (1) to a form

$$y = x\, e^{kx} \tag{11}$$

that by inversion has been seen to produce Lambert W-functions in the case $k = 1$, but trivializes (becomes $x(y) = y$) in the case $k = 0$. I look here to details of the transition

$$0 \xleftarrow[k]{\hspace{4em}} 1$$

—a process of interest because it morphs unfamiliar functions to familiar ones, and entails a loss of double-valuedness.

Graphs of the functions $f(x,k) = x\, e^{kx}$ $(k = \frac{1}{10}n : n = 0, 1, \ldots, 10)$ are shown in Figure 9f. $f(x,0) = x$ is linear, but

$$\lim_{-\infty \leftarrow x} f(x, k > 0) = 0, \text{ with minimum } -1/ke \text{ at } x = -1/k$$

The minimum gets deeper and moves farther to the left as $0 \leftarrow k$.

From $\frac{d}{dy}[x(y) \exp kx(y) - y] = x'(y) \exp kx(y)[kx(y) + 1] - 1 = 0$ we by `DSolve` obtain

$$\begin{aligned} x(y) &= \frac{W(ky)}{k} = \frac{y}{\exp[W(ky)]} \\ &= y - ky^2 + \tfrac{3}{2}k^2y^3 - \tfrac{8}{3}k^3y^4 + \tfrac{125}{24}k^4y^5 - \tfrac{54}{5}k^5y^6 + \tfrac{16384}{315}k^6y^7 - \cdots \\ &= \begin{cases} W(y) & : \quad k = 1 \\ \quad y & : \quad k = 0 \end{cases} \end{aligned}$$

which by Lagrange inversion gives back

$$y(x) = x + kx^2 + \tfrac{1}{2}k^2x^3 + \tfrac{1}{6}k^3x^4 + \tfrac{1}{24}k^4x^5 + \cdots = x\, e^{kx}$$

[16] "Uber die numerische Anflösung transcendenter Gleichungen durch successive Approximationen," Gött. Nchr., December 1898. This is a paper of which Robert Warnock has proposed to prepare an English translation.

$W(ky)/k$ is real for $y > -1/ke$, at which point it assumes its minimum value $-1/k$. We conclude that

$$x(y) = \begin{cases} W_0(ky)/k & : \quad y > -1/ke \\ W_{-1}(ky)/k & : \quad y < -1/ke \end{cases} \tag{12}$$

The transition point moves down and to the left as k decreases, in both cases approaching $-\infty$ as $0 \leftarrow k$, in which limit $W_0(ky)/k \rightarrow y$ and the subordinate branch of $x(y)$ evaporates. Those functions are plotted (for k-values $k = \frac{1}{10}n$: $n = 0, 1, \ldots, 10$) in Figure 9g. The complicated inverse function (12) has morphed to the simple function $x(y) = y$.

Turning now from generalized Lambert inversion to generalized Sommerfeld inversion, graphs of the functions

$$y = f(x, k) = x^k \log x \tag{1}$$

are, for the same assortment of k-values, shown in Figure 10f. $f(x, k)$ assumes its minimal value $-1/ke$ at $x = e^{-1/k}$, which move $\downarrow -\infty$ and $0 \leftarrow$ (respectively) as $0 \leftarrow k$. It is evident from foregoing work that

$$x(y) = \begin{cases} \exp[W_0(ky)/k] & : \quad y > -1/ke \\ \exp[W_{-1}(ky)/k] & : \quad y < -1/ke \end{cases} \tag{13}$$

which produce Figure 10g. At the transition point $y = -1/ke$ the inverse function $x(y)$ assumes the value $e^{-1/k}$, which $\downarrow 0$ as $0 \leftarrow k$. At $k = 0$ the subordinate branch of (13) evaporates, and we are left with $x(y) = e^y$; the complicated function (13)—the generalized Sommerfeld function—has morphed to a simple exponential.

The recursive procedures described on page 13 generalize straightforwardly. When (1) is formulated

$$x = \left(\frac{y}{\log x}\right)^{1/k} = \exp\left(\frac{y}{x^k}\right)$$

we are led to define

$$\begin{aligned} p(y, s, k) &= \left(\frac{y}{\log s}\right)^{1/k} \\ q(y, s, k) &= \exp\left(\frac{y}{s^k}\right) \end{aligned} \tag{14}$$

and, proceding as before, find that—when $0 < k \leqslant 1$, and $y < 0$ falls within the k-specific range described above—$P(y, k, n)$ and $Q(y, k, n)$ reproduce the values assumed by (respectively) the lower/upper branches of (13), and that $Q(y, k, n)$ reproduces the upper branch even when $y > 0$. In the limit $0 \leftarrow k$ the P-process becomes nonsensical (the lower branch has evaporated), but the Q-process yields the anticipated result e^y, irrespective of the sign of y.

I gather that recursive function theory has been carried to a high level by pure/applied "computability" theorists. At a more pedestrian level, I would, with (10) and (14) in mind, like to possess general criteria that—given $f(x,s)$—speak to the convergence of

$$f(x, f(x, f(x, \cdots)))$$

analogous to the criteria that speak to the convergence of infinite sequences (the ultimate name of the game), series and continued fractions.[17]

Complexification. The story acquires complications—complexities that are in some respects surprising/counterintuitive—when the variables $\{x, y\}$ move off the real line onto the complex plane (as Sommerfeld's application required). Those are surveyed—mainly by appeal to 3D displays of the modulus, phase, real & (especially) imaginary parts of the relevant complex functions—in "A paradox involving Sommerfeld's function" (July 2017).[18]

[17] I remark in this connection that we are led by the identity

$$\sqrt{x} = 1 + \frac{x-1}{1+\sqrt{x}}$$

to define

$$f(x,s) = 1 + \frac{x-1}{1+s}$$

```
R[x_,n_]:=NestList[f[y,#]&,y,n]
```

which for any x displays $\sqrt{x}$ as a continued fraction. In the case $x = 2$, $n = 9$ we get

$$\Big\{2, \tfrac{4}{3}, \tfrac{10}{7}, \tfrac{24}{17}, \tfrac{58}{41}, \tfrac{140}{99}, \tfrac{338}{239}, \tfrac{861}{577}, \tfrac{1970}{1393}, \tfrac{4756}{3363} = 1.41421 = \sqrt{2}\,\Big\}$$

[18] The "paradox" of the title—now resolved—referred to the fact that Sommerfeld's P-process does not reproduce the values produced by the analytical result $g(y) = \exp[W(y)]$; to obtain agreement one must, as we have seen, bring $W_{-1}(y)$ into play.